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Note

# On $F$-Hamiltonian graphs

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**Abstract**

Suppose $G$ is a graph, $F$ is a 1-factor of $G$. $G$ is called $F$-Hamiltonian, if there exists a Hamiltonian cycle containing $F$ in $G$. In this paper, two necessary and sufficient conditions for a general graph and a bipartite graph being $F$-Hamiltonian are provided, respectively.


*Keyword:* $F$-Hamiltonian

## 1. Introduction

The definitions in this paper are based on [3]. All the graphs we consider are simple. Suppose $G$ is a graph and $F$ is a 1-factor of $G$, if there exists a Hamiltonian cycle containing $F$ in $G$, then $G$ is called $F$-Hamiltonian. For $V_1 \subseteq V(G)$, $V_2 \subseteq V(G), V_1 \cap V_2 = \emptyset$, let $(V_1, V_2) = \{e \mid e = (x,y) \in E(G), x \in V_1,\ y \in V_2\}, [V_1 : V_2] = |(V_1, V_2)|$. For $S \subseteq V(G)$, $M \subseteq E(G)$, the subgraph $G - S$ is defined to be the subgraph of $G$ obtained by removing all the vertices in $S$ and all edges incident to them, the subgraph $G - M$ is defined to be what is obtained by removing all the edges in $M$ from $G$. $P[x, y]$ means a path whose two ends are $x$ and $y$. Take $K_6$ to be the complete graph with vertex set $V(K_6) = \{y_i \mid 1 \leqslant i \leqslant 6\}$. Let $S_1 = K_6 - \{(y_1, y_2), (y_1, y_4)\}, (y_2, y_3), (y_3, y_4)\}$. Take $K_{3,3}$ to be the complete bipartite graph with bipartition $(A_1, B_1)$, where $A_1 = \{x_1, x_2, x_3\}$; $B_1 = \{y_1, y_2, y_3\}$. Let $S_2 = K_{3,3} - \{(x_1, y_1), (x_2, y_2)\}$. Terminology without explanation here can be seen in [3].

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## 2. Preliminary lemmas

**Lemma 1** (Haggkvist [1]). *Let $G$ be a graph, $|V(G)| = n \geqslant 4$, $n$ is even. If for each pair $u$, $v$ of nonadjacent vertices of $G$, we have*

$$d_G(u) + d_G(v) \geqslant n + 1.$$

*Then, for any 1-factor $F$ of $G$, $G$ is $F$-Hamiltonian.*

**Lemma 2** (Las Vergnas [2]). *Let $G = (A, B, E)$ be a bipartite graph such that $|A| = |B| = n \geqslant 2$. If, for each pair $u, v$ of nonadjacent vertices with $u \in A$, $v \in B$, we have*

$$d_G(u) + d_G(v) \geqslant n + 1.$$

*Then, for any 1-factor of $G$, $G$ is $F$-Hamiltonian.*

## 3. Main results

**Theorem 1.** *Let $G$ be a graph on $n$ ($n \geqslant 4$, $n$ is even) vertices,*

$$\delta(G) = \min_{x \in V(G)} \{d_G(x)\} \geqslant 2, \qquad |E(G)| \geqslant \frac{(n-1)(n-2)}{2} + 1,$$

*then, for any 1-factor of $G$, $G$ is $F$-Hamiltonian if and only if $G \ncong S_1$.*

**Proof.** ($\Rightarrow$) For 1-factor $F = \{(y_1, y_3), (y_2, y_4), (y_5, y_6)\}$ of $S_1$, it is easy to verify that $S_1$ is not $F$-Hamiltonian, hence $G \ncong S_1$.

($\Leftarrow$) For any 1-factor of $G$, when $n = 4$ or $G \cong K_6$, the statement is true. Suppose $n = 6$ and $G \ncong K_6$. For each pair $x, y$ of nonadjacent vertices of $G$, since $|E(G)| \geqslant 11$, then $d_G(u) + d_G(v) \geqslant 5$. Furthermore, if $d_G(u) + d_G(v) \geqslant 7$, $G$ is $F$-Hamiltonian by Lemma 1. Therefore, we may assume that there exist $u, v, (u, v) \notin E(G)$, such that $5 \leqslant d_G(u) + d_G(v) \leqslant 6$. If $d_G(u) + d_G(v) = 5$, then $G - \{u, v\}$ is a complete graph, the statement is clearly true. Otherwise, $G - \{u, v\}$ has at most one pair of nonadjacent vertices. Without loss of generality, we may assume that $G - \{u, v\}$ has exactly one pair of nonadjacent vertices. Suppose $V(G - \{u, v\}) = \{x_1, x_2, x_3, x_4\}$, $(u, x_1) \in F$, $(v, x_2) \in F$ and $(x_3, x_4) \in F$. Consider the following two cases.

*Case* (a): $(x_1, x_2) \notin E(G)$. Since $d_G(u) + d_G(v) = 6$, then $\max\{d_G(u), d_G(v)\} \geqslant 3$, we may assume $d_G(u) \geqslant 3$. It follows that $(u, x_3) \in E(G)$ or $(u, x_4) \in E(G)$. Suppose $(u, x_3) \in V(G)$ (when $(u, x_4) \in E(G)$, the proof is similar).

*Subcase* (a.1): $(v, x_1) \in E(G)$. Then $C = v x_1 u x_3 x_4 x_2 v$ contains $F$.

*Subcase* (a.2): $(v, x_1) \notin E(G)$. Since $d_G(v) \geqslant 2$, we have $(v, x_3) \in E(G)$ or $(v, x_4) \in E(G)$.

*Subcase* (a.2.1): $(v, x_3) \in E(G)$. In this subcase, we must have $(u, x_2) \in E(G)$. If $(u, x_2) \notin E(G)$, then $(u, x_4) \in E(G)$ and $(v, x_4) \in E(G)$, thus $G \cong S_1$. This contradicts the hypothesis. Because of $(u, x_2) \in E(G)$, we obtain $C = v x_3 x_4 x_1 u x_2 v$ containing $F$.

*Subcase* (a.2.2): $(v,x_4)\in E(G)$. It is similar to the method used in (a.2.1).

*Case* (b): $(x_1,x_2)\in E(G)$. Similarly to (a), we may assume that $(u,x_3)\in E(G)$. If $(v,x_4)\in E(G)$, $C=x_4vx_2x_1ux_3x_4$ contains $F$, otherwise $(v,x_4)\notin E(G)$.

*Subcase* (b.1): $d_G(v)=2$. Since $d_G(u)=6-d_G(v)=4$, we have $(u,x_2)\in E(G)$, $(u,x_3)\in E(G)$ and $(u,x_4)\in E(G)$.

*Subcase* (b.1.1): $(v,x_3)\in E(G)$. Then $C=vx_3x_4ux_1x_2v$ contains $F$.

*Subcase* (b.1.2): $(v,x_3)\notin E(G)$. We must have $(v,x_1)\in E(G)$. If $(x_2,x_4)\in E(G)$, then $C=x_3x_4x_2vx_1ux_3$ contains $F$. Otherwise, Since $G-\{u,v\}$ has exactly one pair of non-adjacent vertices, we have $(x_3,x_2)\in E(G)$. If follows that $C=x_4x_3x_2vx_1ux_4$ contains $F$.

*Subcase* (b.2): $d_G(v)>2$. Since $(v,x_4)\notin E(G)$, we have $(v,x_1)\in E(G)$, $(v,x_3)\in E(G)$. If $(x_2,x_4)\in E(G)$, then $C=x_3x_4x_2vx_1ux_3$ contains $F$. Otherwise, we have $(x_2,x_3)\in E(G)$.

*Subcase* (b.2.1): $(u,x_2)\in E(G)$. Then $C=x_3x_4x_1ux_2vx_3$ contains $F$.

*Subcase* (b.2.2): $(u,x_2)\notin E(G)$. Since $d_G(u)\geqslant 3$, we have $(u,x_4)\in E(G)$. It follows that $C=x_3x_4x_1vx_2x_3$ contains $F$.

To sum up, we may assume that $n\geqslant 7$ in what follows. If $G\cong K_n$, of course $G$ is $F$-Hamiltonian. Otherwise, for each pair $x,y$ of nonadjacent vertices, we have

$$d_G(x)+d_G(y)+\frac{(n-2)(n-3)}{2}\geqslant\frac{(n-1)(n-2)}{2}+1.$$

Therefore,

$$d_G(x)+d_G(y)\geqslant n-1.$$

Furthermore, if $d_G(x)+d_G(y)\geqslant n+1$, then $G$ if $F$-Hamiltonian by Lemma 1. Otherwise, there exist $u,v\in V(G)$, $(u,v)\notin E(G)$, such that

$$n-1\leqslant d_G(u)+d_G(v)\leqslant n.$$

Suppose $x\in V(G), y\in V(G)$ such that $(u,x)\in F$, $(v,y)\in F$. Let $G^{(1)}=G-\{u,v\}$, $e_1=(u,x)$, $e_2=(v,y)$, $F_1=F-\{e_1,e_2\}$, $G^{(2)}=G-\{u,v,x\}$, and $G^{(3)}=G-\{u,v,x,y\}$. Consider the following cases.

*Case* (1): $d_G(u)+d_G(v)=n-1$. Then $G^{(1)}$ is complete graph, since $d_G(u)+d_G(v)\geqslant 7$, we may assume that $d_G(u)\geqslant 4$. Hence, there exists $z\in V(G^{(1)})$, $z\neq x$, $z\neq y$, such that $(u,z)\in E(G)$.

*Subcase* (1.1): $(v,x)\in E(G)$. In complete graph $G^{(2)}$, there exists a path $P[y,z]$ containing $F_1$. Then, $C=P[y,z]\ uxvy$ contains $F$.

*Subcase* (1.2): $(v,x)\notin E(G)$. As $d_G(v)\geqslant 2$, we have $w\in V(G^{(1)})$, $w\neq y$, such that $(v,w)\in E(G)$.

*Subcase* (1.2.1): $(u,y)\in E(G)$. It is similar to (1.1).

*Subcase* (1.2.2): $(u,y)\notin E(G)$. Since $G^{(3)}$ is $F_1$-Hamiltonian, without loss of generality, suppose $C=a_1b_1a_2b_2\ldots a_{(n-4)/2}b_{(n-4)/2}a_1$ is a Hamiltonian cycle containing $F_1$

in $G^{(3)}$, and $(a_i,b_i)\in F$, $1\leqslant i\leqslant n-4/2$. Consider the sum

$$\begin{aligned} d_{G^{(3)}}(u)+d_{G^{(3)}}(v) &= \sum_{k=1}^{(n-4)/2}([u:b_k]+[v:a_{k+1}]+[u:a_{k+1}]+[v:b_k]) \\ &= d_G(u)+d_G(v)-2 \\ &= n-3>n-4(a_{(n-4)/2+1}=a_1). \end{aligned}$$

Hence, for some $m$, $1\leqslant m\leqslant (n-4)/2$, we have

$$[u:b_m]+[v:a_{m+1}]+[u:a_{m+1}]+[v:b_m]\geqslant 3.$$

Without loss of generality, assume $m=1$ and $[u:b_1]+[v:a_2]=2$, i.e., $(u,b_1)\in E(G)$ and $(v,a_2)\in E(G)$. So, we have $C=a_1b_1uxyva_2b_2\ldots a_{(n-4)/2}b_{(n-4)/2}a_1$ containing $F$.

*Case* (2): $d_G(u)+d_G(v)=n$. Then, $G^{(1)}$ has at most one pair of nonadjacent vertices. When $G^{(1)}\cong K_{n-2}$, the proof is similar to (1). We may therefore assume $G^{(1)}$ has exactly one pair of nonadjacent vertices.

*Subcase* (2.1): $(x,y)\notin E(G)$.

*Subcase* (2.1.1): $(v,x)\in E(G)$. It is similar to (1.1).

*Subcase* (2.1.2): $(v,x)\notin E(G)$. Then, there exists $w\in V(G^{(1)})$, $w\neq y$, such that $(v,w)\in E(G)$.

*Subcase* (2.1.2.1): $(u,y)\in E(G)$. It is similar to (1.2.1).

*Subcase* (2.1.2.2): $(u,y)\notin E(G)$. Then, $d_{G^{(3)}}(u)+d_{G^{(3)}}(v)=n-2$. Since $G^{(3)}\cong K_{n-4}$, without loss of generality assume that $C=a_1b_1a_2b_2\ldots a_{(n-4)/2}b_{(n-4)/2}a_1$ is a Hamilton cycle in $G^{(3)}$ and $(a_i,b_i)\in F$, $1\leqslant i\leqslant (n-4)/2$. Consider the following inequalities.

$$\begin{aligned} d_{G^{(3)}}(u)+d_{G^{(3)}}(v) &= \sum_{k=1}^{(n-4)/2}([u:a_k]+[v:b_k]+[u:b_k]+[v:a_k]) \\ &= n-2>n-4. \end{aligned}$$

Therefore, there exists an $m\in\{1,2,\ldots,\frac{(n-4)}{2}\}$, such that

$$[u:a_m]+[v:b_m]+[u:b_m]+[v:a_m]\geqslant 3.$$

We may assume that $m=1$ and $[u:a_1]+[v:b_1]=2$, i.e.,

$$(u,a_1)\in E(G),\ (v,b_1)\in E(G).$$

Let $e_3=(a_1,b_1)$, $G^{(4)}=G-\{u,v,a_1,b_1\}$. Since $G^{(4)}$ has only one pair $x,y$ of nonadjacent vertices, we find a path $P[y,x]$ containing $F-\{e_1,e_2,e_3\}$. Thus, $C=P[y,x]ua_1b_1vy$ contains $F$.

*Subcase* (2.2): $(x,y)\in E(G)$.

*Subcase* (2.2.1): $(u,y)\in E(G)$, $(v,x)\in E(G)$. Then $d_{G^{(3)}}(u)+d_{G^{(3)}}(v)=n-4\geqslant 4$. We may assume that $d_{G^{(3)}}(u)\geqslant 2$, i.e., $u$ are adjacent with at least two vertices, say $a_p,a_q$ of $G^{(3)}$. By Lemma 1, $G^{(3)}$ is $F_1$-Hamiltonian. Without loss of generality, suppose $C=a_1a_2a_3\ldots a_{n-4}a_1$ is a Hamiltonian cycle which contains $F_1$ in $G^{(3)}$, and

$(a_i, a_{i+1}) \in F$ for $i = 1, 3, 5, \ldots, n-5$. The vertices that are not adjacent in $G^{(3)}$ are $p_1$ and $p_2$ with $a_p \neq a_{n-4}$.

*Subcase* (2.2.1.1): $y \in \{p_1, p_2\}$. Without loss of generality, suppose $(y, a_1) \notin E(G)$. When $p$ is even, $C = a_p uxvya_{p+1} \ldots a_{n-5}a_{n-4}a_1a_2 \ldots a_p$ is a Hamiltonian cycle containing $F$. When $p$ is odd, $C = a_{p-1}a_{p-2} \ldots a_1 a_{n-4}a_{n-5} \ldots a_p uxvya_{p-1}$ contains $F$.

*Subcase* (2.2.1.2): $y \notin \{p_1, p_2\}$. Then $y$ is adjacent to each vertex of $G^{(3)}$. It is easy to construct a Hamiltonian cycle which contains $F$.

*Subcase* (2.2.2): $(u, y) \notin E(G)$ or $(v, x) \notin E(G)$. Then, $d_{G^{(3)}}(u) + d_{G^{(3)}}(v) > n-4$. Therefore, there is $m \in \{2, 4, 6, \ldots, n-4\}$ such that

$$[u : a_m] + [v : a_{m+1}] + [u : a_{m+1}] + [v : a_m] \geqslant 3.$$

We may assume

$$[u : a_m] + [v : a_{m+1}] = 2,$$

i.e.,

$$(u, a_m) \in E(G) \text{ and } (v, a_{m+1}) \in E(G).$$

Thus, $C = a_m a_{m-1} \ldots a_1 a_{n-4} a_{n-5} \ldots a_{m+1} vyxua_m$ contains $F$.

The proof of Theorem 1 is complete. □

**Theorem 2.** *Let $G$ be a bipartite graph with bipartition $(A, B)$. Such that $|A| = |B| = n \geqslant 2$, $\delta(G) = \min_{x \in V(G)}\{d_G(x)\} \geqslant 2, |E(G)| \geqslant n^2 - n + 1$. Then, for any 1-factor $F$ of $G$, $G$ is $F$-Hamiltonian if and only if $G \ncong S_2$.*

**Proof.** ($\Rightarrow$) For 1-factor $F = \{x_1, y_2\}, (y_1, x_2), (x_3, y_3)\}$ of $S_2$, since $S_2$ is not $F$-Hamiltonian, we have $G \ncong S_2$.

($\Leftarrow$) For bipartite graph $G$ with bipartition $(A, B)$, suppose $A = \{x_1, x_2, \ldots, x_n\}$; $B = \{y_1, y_2, \ldots, y_n\}$ and $F$ is a 1-factor of $G$. If $G$ is a complete bipartite graph, the statement is clearly true. Otherwise, for each pair $x$, $y$ of nonadjacent vertices with $x \in A$, $y \in B$, we have

$$d_G(x) + d_G(y) \geqslant n.$$

Furthermore, if $d_G(x) + d_G(y) \geqslant n + 2$, then $G$ is $F$-Hamiltonian by Lemma 2. Otherwise, there exist $u, v \in V(G)$, $u \in A$, $v \in B$, $(u, v) \notin E(G)$, such that

$$n \leqslant d_G(x) + d_G(y) \leqslant n + 1.$$

Choose $x_m \in A$, $y_l \in B$, $x_m \neq u$, $y_l \neq v$, such that $(u, y_l) \in F$ and $(v, x_m) \in F$, $2 \leqslant m, l \leqslant n$. Without loss of generality, we may assume that $u = x_1$, $v = y_1$, $x_m = x_2$, $y_l = y_2$. Put $G^{(*)} = G - \{x_1, y_1\}$, $G' = G - \{x_1, y_1, x_2, y_2\}$. $e_1 = (x_1, y_2)$, $e_2 = (y_1, x_2)$. Consider the following cases.

*Case* (1): $d_G(x_1) + d_G(y_1) = n$. Then, $G^{(*)}$ is a complete bipartite graph. Since $\delta(G) \geqslant 2$, it is allowed to choose $x_p \in A$, $y_q \in B$ such that $(x_1, y_q) \in E(G)$ and $(y_1, x_p) \in E(G)$, $3 \leqslant p, q \leqslant n$, and let $e_3 = (x_p, y_q)$.

*Subcase* (1.1): $e_3 \in F$. In the complete bipartite graph $G - \{x_1, y_1, x_p, y_q\}$, there exists a path $P[y_2, x_2]$ which contains $F - \{e_1, e_2, e_3\}$. Then, $C = P[y_2, x_2]y_1x_py_qx_1y_2$ contains $F$.

*Subcase* (1.2): $e_3 \notin F$. In the complete bipartite graph $G'$, there exists a path $p[y_q, x_p]$ which contains $F - \{e_1, e_2\}$. Then, $C = P[y_q, x_p]y_1x_2y_2x_1y_q$ contains $F$.

*Case* (2): $d_G(x_1) + d_G(y_1) = n + 1$. Then, $G^{(*)}$ has at most one pair of nonadjacent vertices. Without loss of generality, we assume that $G^{(*)}$ has exactly one pair of nonadjacent vertices. In this case, we must have $n \geqslant 4$. If $n = 3$, then $d_G(x_1) + d_G(y_1) = 4$, $E(G^{(*)}) = 3$. Hence $G \cong S_2$. A contradiction appears. Now, suppose $n \geqslant 4$. Since $d_G(x_1) + d_G(y_1) > n - 2$, there exist $x_{p_1} \in V(G'), y_{q_1} \in V(G')$, such that $(x_1, y_{q_1}) \in E(G)$, $(y_1, x_{p_1}) \in E(G)$, $(x_{p_1}, y_{q_1}) \in F$. Without loss of generality, suppose $x_{p_1} = x_3$, $y_{q_1} = y_3$, $e' = (x_3, y_3)$.

*Subcase* (2.1): $(x_2, y_2) \notin E(G)$. In the bipartite graph $G - \{x_1, y_1, x_3, y_3\}$, there exists a path $P[x_2, y_2]$ which contains $F - \{e_1, e_2, e'\}$, and hence $C = P[x_2, y_2]x_1y_3x_3y_1x_2$ contains $F$.

*Subcase* (2.2): $(x_2, y_2) \in E(G)$. Without loss of generality, assume that $(x_n, y_n) \notin E$ and $(x_3, y_3) \in F$, $(x_4, y_4) \in F, \ldots, (x_{n-2}, y_{n-2}) \in F$, $(x_{n-1}, y_n) \in F$ and $(y_{n-1}, x_n) \in F$. We obtain $C = y_2x_1y_3x_3y_1x_2y_nx_{n-1}y_{n-1}x_ny_{n-2}x_{n-2} \ldots y_4x_4y_2$ is a Hamiltonian cycle containing $F$.

The proof of Theorem 2 is complete. □

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